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Lower Deviation Probability for the Maximum of a Branching Random Walk

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Abstract

Given a supercritical branching random walk, considerting its maximal position, Gantert and Höfelsauer (2018) and Chen and He (2020) obtained the first order of the decay for the deviation probabilities. In this work, we get a preciser result for the decay of the lower deviation probability, i.e. $\mathbb{P}(M_n \leq \alpha x^* n)$ for $\alpha < 1$ in Schröder case.

Introduction

Branching random walk and its maximum

We study the maximum of a branching random walk in discrete-time on $\mathbb{R}$, which is a generalized branching process. Generally, to construct a branching random walk, we take a random point measure to describe both branching and motions. Each individual produces independently its children according to the law of this random point measure. In this work, we study a relatively simpler model which is constructed as follows. We take a Galton–Watson tree $\mathcal{T}$, rooted at ρ, with offspring distribution given by $\{p_k\}_{k\geq 0}$. For any $u, v \in \mathcal{T}$, we write $u \preceq v$ if u is an ancestor of v or $u = v$. Moreover, to each node $v \in \mathcal{T}\backslash\{\rho\}$, we attach a real-valued random variable X_v to represent its displacement. So the position of v is defined by

$$S_v := \sum_{\rho \prec u \preceq v} X_u.$$

Suppose that given the tree $\mathcal{T}$, $\{X_v; v \in \mathcal{T}\backslash\{\rho\}\}$ are i.i.d. copies of some random variable X (which is called displacement or step size). Let $S_\rho := 0$ for convenience, $|u|$ denote the generation of an individual $u \in \mathcal{T}$, i.e. the graph distance between ρ and u. Thus, $\{S_u; u \in \mathcal{T}\}$ is our branching random walk with independence between offsprings and motions. This independence will be necessary for our arguments. The maximum of the branching random walk at the nth generation is defined as

$$M_n := \sup_{|v|=n} S_v.$$

In fact, branching random walk in the Schröder case can be viewed as a generalized version of branching Brownian motion. For maximum of branching Brownian motion, Chauvin and Rouault [1] first investigated the large deviation probabilities. Derrida and Shi [4] studied the large deviation and obtained the exponential decay. Later, Derrida and Shi [5] conjectured the second order of the decay for the deviation probabilities. They established precise estimates. Chen and He [3]verified their conjecture [5] and obtained the asymptotic of $\mathbb{P}(M_t \leq \sqrt{2}\alpha t)$ for $\alpha_c = 1 - \sqrt{2}$. The lower deviation probability $\mathbb{P}(M_t \leq \sqrt{2}\alpha t)$ exhibits a phase transition, rate function is not analytic, at α_c.

For maximum of branching random walks, Gantert and Höfelsauer [6] proved that in the Schröder case $\mathbb{P}(M_n \leq xn)$ decays exponentially. Chen and He [2] also obtained the lower deviation of M_n, i.e. $\mathbb{P}(M_n \leq xn)$ for $x < x^*$ in the Böttcher case, which completes the work [6], where $x^* = \sup\{x \geq 0 : I(x) \leq \log m\} \in (0, \infty)$.

Motivated by [2][5] and [6], we study a preciser result for the decay of the lower deviation probability, i.e. $\mathbb{P}(M_n \leq \alpha x^* n)$ for $\alpha < 1$ in Schröder case. We are interested in the supercritical case where $m := \sum_{k\geq 0} kp_k > 1$ and the system survives with positive probability. Let $\{S_n\}_{n\geq 0}$ be a random walk started from 0 with i.i.d. increments distributed as X, a Gaussian random variable. Observe that for any individual u of the nth generation, S_u is distributed as S_n. For $x \in \mathbb{R}$, the rate function of the random walk $\{S_n\}_{n\geq 0}$ is defined as

$$I(x) = \sup_{\lambda\in\mathbb{R}} \left(\lambda x - \log \mathbb{E}\left[e^{\lambda X}\right]\right).$$

Define the rate function Ψ of the branching random walk as

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}(M_n \leq xn) = \Psi(x) = \begin{cases} I(x) - \log m, & \text{for } x > x^*; \\ 0, & \text{for } x = x^*; \\ H(x), & \text{for } x < x^*, \end{cases}$$

where $H(x) = \inf_{t\in(0,1]} \left\{t\rho + tI\left(t^{-1}(x - (1-t)x^*)\right)\right\}$. Under some assumptions, it is proved ([6] Theorem 3.2) that the laws of $\frac{M_n}{n}$ satisfy a large deviation principle with rate function $\Psi(x)$. The rate function Ψ is given explicitly by Chen and He [2]. It is worth mentioning that this rate function Ψ also is not analytic at the point $x_0 (x_0 < x^*)$, as shown in Figure 1, where C_1 is a negative constant and $\rho := -\log \mathbb{E}\left[Z_1 q^{Z_1 - 1}\right] \in (0, \infty]$.

Main results

In our work, for convenience, let $\alpha := \frac{x}{x^*} < 1$ and $\gamma := \frac{x_0}{x^*}$. For $1 > \alpha > \gamma$, $\alpha = \gamma$ and $\alpha < \gamma$, the lower deviation probability is divided respectively into supercritical, critical and subcritical case. This effect is caused by the fact that the optimal strategy leading to lower deviations depends on whether $\alpha < \gamma$ or $\alpha > \gamma$. Let us introduce τ defined by

$$\tau := \inf\{n \geq 0 : Z_n \geq 2\},$$

where Z_n denotes the number of particles at the nth generation. The large deviation event $\{M_n \leq \alpha x^* n\}$ for $\alpha < 1$ is governed by the value of τ. Take $p_0 = 0$, $p_1 + p_2 = 1$, we have

$$\mathbb{P}(M_n \leq \alpha x^* n) = e^{n(\log p_1)} \mathbb{P}(S_n \leq \alpha x^* n) + \sum_{i=0}^{n-1} p_1^i p_2 \int_{\mathbb{R}} \mathbb{P}(S_i \in \mathrm{d}y)(\mathbb{P}(M_{n-i} \leq \alpha x^* n - y))^2.$$

The first term on the right-hand side comes from the event $\{\tau > n\}$ on which an ancestor has only one child, the second term is obtained by applying Markov property at the time τ. We have obtained the result in the subcritical case, but the critical and supercritical cases are being calculated.

Theorem 1. *If $\alpha < \gamma$, as $n \to \infty$, we have*

$$\lim_{n\to\infty} n^{\frac{1}{2}} e^{((\alpha x^*)^2/2 - \log p_1)n} \mathbb{P}(M_n \leq \alpha x^* n) = \frac{C(\alpha)}{\sqrt{2\pi}},$$

where $C(\alpha) =: -\frac{1}{\alpha x^*} + \sum_{j=1}^{\infty} e^{-((\alpha x^*)^2/2 + \log p_1)j} \int_{\mathbb{R}} e^{\alpha x^* z} (\mathbb{P}(M_j \leq z))^2 \mathrm{d}z < \infty.$

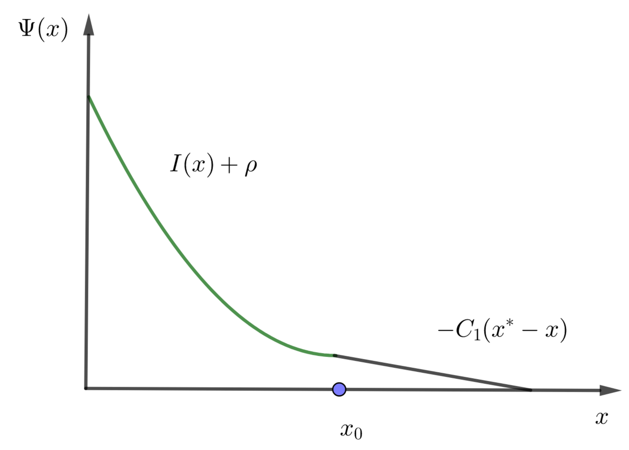

Figure 1: Rate Function $x < x^*$

Further Research

One may instead apply the arguments in the Böttcher case. We could also consider the precise asymptotic behaviour of the corresponding upper deviation probability $\mathbb{P}(M_n \geq \alpha x^* n)$ for $\alpha > 1$ in the Böttcher case and Schröder case.

In this paper, we study the branching random walk by assuming that the branching and the motions are independent. However, things will be more complicated in the general setting where the reproduction law is given by a point process representing the displacements of children of one individual.

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